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Multi-scaling mix and non-universality between population and facility density

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ABSTRACT

The distribution of facilities is closely related to our social economic activities. Recent studies have reported a scaling relation between population and facility density, with the exponent depending on the type of facility. In this paper, we show that generally this exponent is not universal for a specific type of facility. Instead, by using Chinese data, we find that it increases with per capita gross domestic product (GDP). Thus our observed scaling law is actually a mixture of several multi-scaling relations. This result indicates that facilities may change their public or commercial attributes according to the outside environment. We argue that this phenomenon results from an unbalanced regional economic level, and suggest a modification of a previous model by introducing the consuming capacity. The modified model reproduces most of our observed properties.

1. Introduction

Facilities and infrastructures such as hospitals, schools, Internet routers, originate from the development of human social civilization and in turn shape our modern daily life. This complex evolution process raises interesting questions of how these facilities are distributed and how they correlate with the overall social economic system. A better understanding of this issue could help to provide better public service and to save social opportunity cost [1]. It is believed that population density and economic factors are crucial in deciding the locations of these facilities [2–9]. But unevenly distributed populations and economic levels make the question very complicated. Although studies ranging from business economics, system engineering, computer science, and geography even to biology have addressed the issue, both the theoretical basis and empirical demonstrations are inadequate [2–12]. This causes the arbitrary assumption of uniformly distributed nodes in many spatial network models, even though this is far from the reality [13,14].

Intuitively, the number of facilities in an area increases with the corresponding population. The related studies can be traced back to a so-called *p-median* problem which aims to find the precise locations of facilities so that the mean distance for one to reach the nearest facility is minimized [15,9]. Numerical and analytic treatments have been used to suggest a relation $D \sim \rho^{\alpha}$, where D is the facility density, ρ is the population density, and $\alpha = 2/3$ [8,4,9,5]. This result accounts for Internet router distribution and territorial divisions but does not agree with other empirical studies in which, although a universal scaling $D \sim \rho^{\alpha}$ is evidenced, the value of α, depending on the type of facility, ranges from 2/3 to 1 rather than just being fixed at 2/3 [2,5,1]. More specifically, a commercial facility and a public one have $\alpha = 1$ and $\alpha = 2/3$, respectively, while for a facility with both attributes, the exponent lies in between. These findings stimulated a recent study in which a general model based on the tradeoff between the profit (commercial concern) and the social opportunity cost (public

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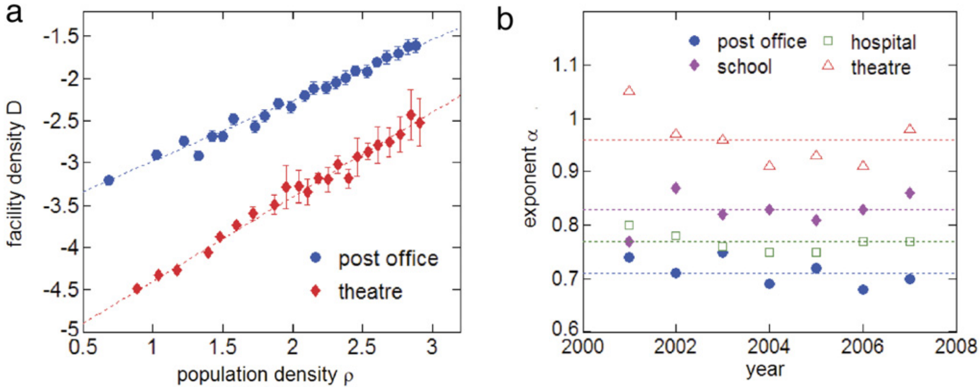

Fig. 1. (a) The scaling law between facility density (D) and population density (ρ) for the cases of post office and theater in 2007. The data are logarithmic binned and are plotted on a log–log scale. The dotted lines are the corresponding fits, which are measured as $D \sim \rho^{0.7}$ (blue) and $D \sim \rho^{0.98}$ (red). (b) The yearly scaling exponents for all the four facilities. Dotted lines are their corresponding averages calculated as 0.71, 0.96, 0.83, 0.77 for post office, theater, school, and hospital, respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

service) was proposed [1]. The value of α in that model represents the type of facility, which is described by the relative weight of commercial and public attributes. In other words, it assumes a universal scaling exponent for a specific type of facility. (Actually, all the previous studies tacitly make this assumption.) Thus if we choose a part of these facilities according to some other properties, the scaling exponent is expected to be unchanged in this sample.

As we will show in this paper, this is not always true. At least in Chinese cases, samples from a specific type of facility according to per capita gross domestic product (GDP) yield an increasing scaling exponent. Therefore the macroscopic power-law relation between facility and population density is actually a mixture of different scaling functions. This multi-scaling property indicates a different picture, namely that the attribute of a facility changes with the outside environment, and consequently this affects its real distribution. In the next section, we will provide empirical evidence for the above arguments. And in Section 3, we will try to offer an explanation and present a modified model to reproduce the multi-scaling property.

2. Empirical study

We have gathered 7-year empirical data (2001–2007) including the positions of four typical types of facility (hospital, post office, school, and theater), the population, GDP, and the area of every county. Despite the temporal fluctuations, there are in total more than 287 counties and about 60,500 hospitals, 56,300 post offices, 309,200 schools and 5000 theaters. All these data come from the CHINA CITY STATISTICAL YEARBOOK, whose electronic versions can be found and downloaded at http://ishare.iask.sina.com.cn. These data allow us to calculate the population density, facility density, per capita GDP, and their correlation at county level. Such coarse-grained treatment was also applied in previous studies due to the resolution limitation of the facility positions [1,2].

Our first observation is consistent with the previous studies. Indeed a scaling relation between population and facility density emerges in all four types of facility (Fig. 1(a)). A detailed analysis of their scaling exponents indicates that, despite the yearly fluctuations, the exponents stay around their own averages, which are measured as 0.71, 0.96, 0.83, 0.77 for post office, theater, school, and hospital, respectively. According to Ref. [1], post office and theater, whose scaling exponents are close to 2/3 and 1, can be viewed as representatives of public and commercial facilities. This is consistent with our experience that a post office disregarding profit is necessary everywhere while a theater behaves oppositely.

To examine the universality assumption of the scaling exponent, we classify the counties into four different classes. Specifically, we calculate the per capita GDP for each county and divide their logarithmic values into four equal intervals. Those counties whose per capita GDP lie in a common interval belong to the same class. Consequently, counties in class 1 have the lowest per capita GDP while those in class 4 have the highest. This classification distributes about 40 counties in class 1 and over 70 in each of the other three classes. Although the number of counties in class 1 is significantly lower than the numbers in the other classes, the number of corresponding facilities is still large enough to apply statistical analysis. Another problem is whether the per capita GDP correlates obviously with the population density such that our method might cause serious statistical bias. This possibility is basically ruled out, as the correlation turns out to be very weak (correlation coefficient < 0.25).

For each type of facility, we study their scaling relation in different classes. As illustrated in Fig. 2, in each class the relation between population and facility density is still a power law. However, their scaling exponents are not equal, but increase clearly with the per capita GDP level (class number). This result contrasts with the universality assumption of the scaling exponent in Ref. [1]. Instead it indicates that the observed scaling relation $D \sim \rho^{\alpha}$ is actually composed of several types of multi-scaling behavior. If we accept the physical meaning of α interpreted in Ref. [1], the multi-scaling phenomenon reveals

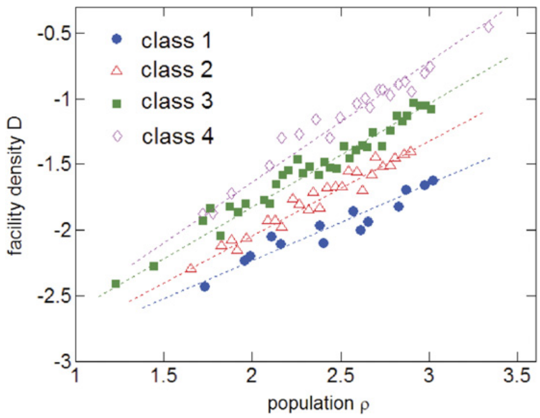

Fig. 2. Multi-scaling relation for the case of hospital in 2007. The class number (1 to 4) describes the increase of the per capita GDP level. For each class, the data are logarithmic binned and are plotted on a log–log scale. Dotted lines are their corresponding fits plotted here as guides for the eye. The data for different classes are shifted a little for better visualization. For each class, D and ρ still follow a power law, but the exponent increases clearly with increasing class number. The result indicates a multi-scaling picture rather than universality of the scaling exponent.

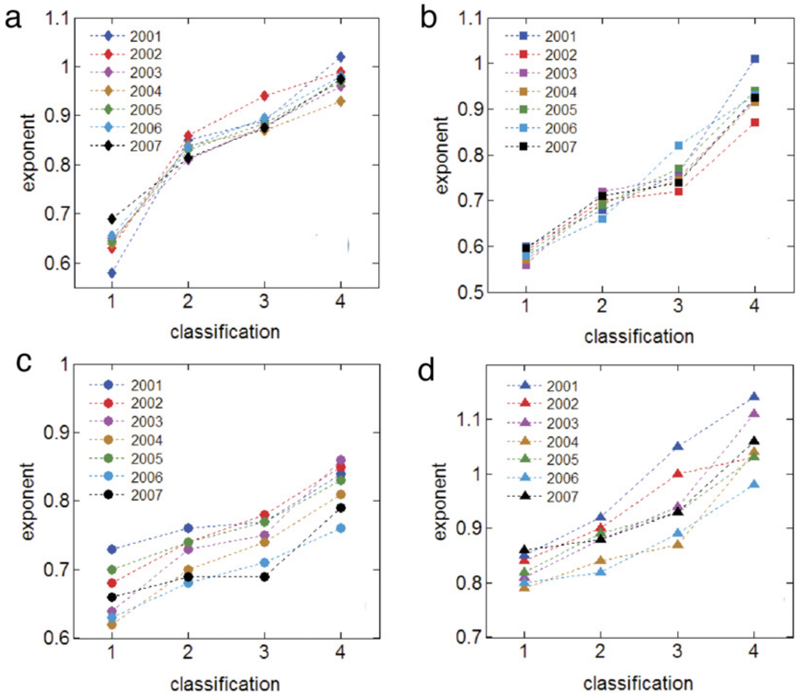

Fig. 3. The relation between the multi-scaling exponents (y-axis) and the per capita GDP level (class number or classification as labeled on x-axis) for four types of facility: (a) school, (b) hospital, (c) post office, and (d) theater. For all four types of facility the exponents increase with increasing per capita GDP every year without exception. For school and hospital, the exponents vary in a large range from 0.6 to 1. In contrast, the exponents of post office and theater are in a narrow range.

an interesting fact that facilities can change their attributes according to the outside economic environment. In particular, they tend to be commercial in high per capita GDP level areas but still provide necessary public services in poorly developed places. Further detailed analysis suggests that this multi-scaling property as well as the positive correlation between the multi-scaling exponents and per capita GDP level occurs every year in all types of facility, regardless of their temporal fluctuations (Fig. 3). For school and hospital (Fig. 3(a) and (b)), the exponents vary from 2/3 to 1, which covers almost the whole possible range. In contrast, the exponents for post office are much more stable. As shown in Fig. 3(c), they stay near 0.7 and increase to no more than 0.85. Similarly, in Fig. 3(d), the range of the exponents for theater is narrow, centering around 0.9. It seems that purely commercial or public facilities are not likely to exhibit diverse attributes, which is somewhat consistent with our intuition.

3. Explanation and model modification

All these findings are not captured by previous models, and thus we require a more complete theory. As will be presented later, a small modification by introducing the consuming capacity can reproduce most of these properties. Before that, we will first review the model of Ref. [1] and clarify some useful concepts.

Suppose that we are given the total population N_p and the total number of facilities N_f on a plane. And suppose people always visit their nearest facility, by which we can define the Voronoi cell V_i, whose area is s_i, as the set of points closer to the ith facility than to any others [9]. This means that the number of visitors to the ith facility is the number of people living in V_i, which is denoted as n_i. Therefore the population density in V_i is calculated as $\rho_i = n_i/s_i$, while the corresponding facility density D_i is $D_i = 1/s_i$.

For a commercial facility, Ref. [1] assumes that the profit of the ith facility is proportional to n_i. Therefore, a facility having lower n_i is better off moving to other location with higher population to obtain a higher profit. This strategy is applied by all facilities during a relocation process. Finally, the system will reach an equilibrium such that every facility has almost the same profit, i.e. $n_i \sim N_p/N_f$. Then by using the expression for ρ_i and D_i calculated above, we arrive at $D \sim \rho$. On the other hand, public facility concerns prior the social opportunity cost caused by the distance between visitors and facilities, which is described by $n_i\langle r_i\rangle$ with $\langle r_i\rangle \sim \sqrt{s_i}$ representing the average distance to the ith facility. To provide better public service, facilities at lower-cost places should be relocated to those with higher $n_i\langle r_i\rangle$. Then, in the steady state, $n_i\langle r_i\rangle$ becomes the same for all facilities. Again by using the expression for ρ_i and D_i, we have $D \sim \rho^{2/3}$. For facilities with both attributes, Ref. [1] defines a general quantity,

$$c_i = n_i\langle r_i\rangle^{\beta}, \tag{1}$$

where β is tunable within the range [0, 1]. Analogously to the above analysis, one can derive the final scaling relation $D \sim \rho^{2/(\beta+2)}$, which gives $\alpha = 2/(\beta + 2)$.

However, in this nicely compact model, the assumption that profit is equal to population is too simple. After all, the profit is closely related to the commodity prices and people's consuming capacity. Facilities providing luxury commodities are not opened in poor places even though they are densely populated because no one there can bear such high-level consumption. On the other hand, facilities in regions with lower population but higher consuming capacity can still benefit from high prices. Therefore facilities with both attributes, due to their commercial component, take the opportunity to locate and profit in developed areas, even though the service there provided by their public component is already adequate. This causes the number of facilities in a well-developed area to be larger than those in poor places even if they are equally populated, which indicates a more rapid increase in number of facilities in high economic-level places. If we assume that the consuming capacity has a positive correlation with per capita GDP, which seems plausible, the above explanation for multi-scaling does indeed make sense.

To make the explanation more convincing, it is useful to introduce an alternative expression of Eq. (1) to characterize the transition from public to commercial facility, expressed as

$$c_i = \lambda n_i + (1 - \lambda)n_i\langle r_i\rangle, \tag{2}$$

where $\lambda \in [0, 1]$ is a tunable parameter controlling the relative weight of commercial or public attributes and consequently determining the final exponent α, just as the role of β in Eq. (1). Eq. (2) has the same physical meaning and similar effect to Eq. (1), but it turns out to be more difficult to apply an analytical treatment with it. However, one can still prove its scaling property by the method used in Ref. [16]. To introduce the consuming capacity to the model, we denote m_i as the average expense consumed by every person living in Voronoi cell V_i so that the profit is equal to $m_i n_i$. Then Eq. (2) is rewritten as

$$c_i = \lambda m_i n_i + (1 - \lambda)n_i\langle r_i\rangle. \tag{3}$$

The only modification compared to Eq. (2) lies in the first term of the right-hand side of Eq. (3), i.e. $\lambda \rightarrow \lambda m_i$. This modification does not affect the macroscopic scaling law between D and ρ qualitatively, but it changes the relative weight of commercial attributes on a microscopic level. In particular, for large m_i, the first term $\lambda m_i n_i$ is enhanced by the effective weight λm_i, which causes the system to be more concerned with profit. This leads the facility to be more commercial and the scaling exponent to be close to 1. On the other hand, if m_i is small, the second term $(\lambda - 1)n_i\langle r_i\rangle$ takes over, and then the system is more concerned with social opportunity cost just like a public facility, and the exponent becomes close to 2/3. Therefore even for the same λ (i.e. the same type of facility), different m_i leads to different system behavior. This is the reason why multi-scaling emerges and why their exponents increase with increasing economic level. Moreover, if $\lambda = 0$ (purely public facility), Eq. (3) becomes independent of m_i. So the system degenerates to the classical model in Ref. [1], and thus displays only a single scaling relation with the exponent $\alpha = 2/3$. On the other hand, if $\lambda = 1$ (purely commercial facility), Eq. (3) depends totally on the first term. But m_i in this case has no effect on the exponent, and only changes the coefficient of the scaling relation, leaving the only exponent $\alpha = 1$, which is also independent of m_i [17]. Therefore the multi-scaling property can be less pronounced in a very commercial or public facility, which explains the narrow range of multi-scaling exponents observed for post office and theater.

If we still adopt the idea in Ref. [1] and follow the expression of Eq. (1), the above explanation indicates that the exponent β in Eq. (1) should be modified to be a function of parameter m_i, i.e. $c_i = n_i\langle r_i\rangle^{\beta(m_i)}$. Then the multi-scaling exponent can

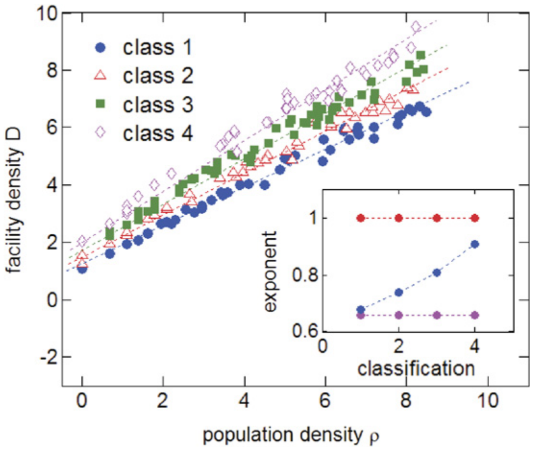

Fig. 4. Multi-scaling relation between D and ρ simulated by the modified model with $\mu = 1$. The data are plotted on a log–log scale. The class number (1 to 4) describes the increase of the level of m. The relation in each class is a power law, but the exponent increases with increasing class number, just as in Fig. 2. Dotted lines are their corresponding fits plotted here as guides for the eye. Inset: the simulated multi-scaling exponent versus class number. For $\mu = 0$ (red) and $\mu \to \infty$ (purple), which represent the pure commercial and public facility, the exponents stabilize at 1 and 2/3, respectively. For intermediate μ, such as $\mu = 1$ (blue), the exponent increases with the classification. The result is averaged over 50 simulations. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

be calculated as $2/(\beta(m_i) + 2)$, and the macroscopic exponent α can be given by taking an average over all possible m_i. To reproduce the multi-scaling property by simulation, we apply the following simple formula for $\beta(m)$:

$$\beta(m) = 1 - (m/m_{\max})^{\mu}, \tag{4}$$

where $m_{\max}$ represents the possible maximum of m and μ is a parameter controlling the sensitivity of β with m. If $\mu = 0$, the facility becomes purely commercial, while, if $\mu \to \infty$, it becomes purely public. So μ also controls the exponent α. The real-world $\beta(m)$ can be quite different (probably related to the type of facility). Finding its precise expression could be a complicated task, and is beyond the scope of this paper. The aim of our simulation is only to reproduce the multi-scaling phenomenon qualitatively. Our simulation follows a similar process to that of Ref. [1], except for the above-proposed modification. Specifically, we first distribute the population density ρ and the economic level m randomly on a plane. And then we further put some facilities of a certain type. At each time step, every facility calculates its benefit according to $c_i = n_i\langle r_i\rangle^{\beta(m_i)}$. The facility with the lowest c_i then moves to the region with the highest benefit. This process is repeated until the system is steady. At steady state, we measure various quantities and their relations, just as for the empirical data. Note that the classification here is carried out according to the value of m. Other details of the simulation such as parameter setting are described in Appendix A. In Fig. 4, we plot the simulated relation between population and facility density for different classes (i.e. different level of m or say different intervals of m). Clearly it displays a multi-scaling property with the exponents increasing with increasing class number. The inset of Fig. 4 presents the results for the simulated multi-scaling exponents. For purely commercial ($\mu = 0$) or purely public ($\mu \to \infty$) facilities, the exponents stabilize at 1 or 2/3. But for intermediate $\mu = 1$, the exponents increase. In the inset of Fig. 5 we demonstrate that the overall scaling relation between D and ρ is maintained in our modified model. And the scaling exponent α in the simulation decreases on increasing the parameter μ, as shown in Fig. 5. Note that all these results can be calculated analytically. We present an analytical solution of $\alpha(\mu)$ in Fig. 5, which is in good agreement with the simulation. More details about the analytical calculations are reported in Appendix B.

4. Conclusion

We have analyzed the scaling relation between population and Chinese facility density at different per capita GDP levels. Our study does not reveal the universality of the scaling exponent but instead suggests a multi-scaling picture. More interestingly, such multi-scaling exponents increase with increasing per capita GDP regardless of the type of facility. These results indicate that facilities can change their commercial or public attributes according to the outside environment, i.e. they take th opportunity to gain more profit in developed areas but still fulfill their public-service responsibilities in poor regions. We have also provided a possible explanation and suggest a modification by considering the consuming capacity. The modified model can reproduce most of our observed properties.

Our study stresses existence rather than universality. Indeed the multi-scaling property is observed every year for all four types of facility. Therefore their occurrence is certain rather than coincidental. On the other hand, it is also appealing to explore whether this phenomenon occurs in other more developed countries, where the economic system is more balanced

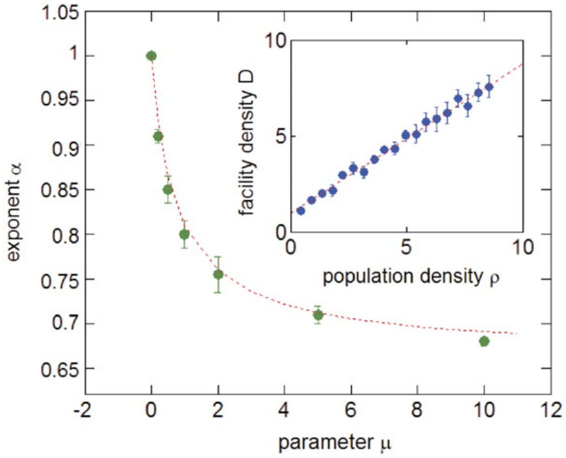

Fig. 5. The relation between the scaling exponent α and the parameter μ. The green points are the simulation result, which is averaged over 50 realizations. The red dotted lines are the analytical prediction. Inset: the scaling relation between population and facility density simulated by our modified model with $\mu = 1$. The simulation data are logarithmic binned and are plotted on a log–log scale. The red dotted line is the fit measured as $D \sim \rho^{0.8}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

and stable. Due to the limitation of data, the present study cannot cover this aspect. If this point is evidenced, it indicates the multi-scaling could be a common property. Otherwise it either requires explanations from sociocultural, economic, political, or other aspects, or leads us to a long-time dynamic evolution picture of facility allocation, both of which are significant for understanding our socioeconomic system or even providing guidelines for urban development.

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Appendix A. Simulation details

We use a coarse-grained simulation for our modified model. We first divide the plane into many unit squares whose areas all equal 1, and assume that all the situations in one unit square are approximately identical. Then we distribute for each unit square u the population ρ_u and the consuming capacity m_u according to the corresponding distribution $p(\rho)$ and $p(m)$. And we further distribute randomly the initial number of facilities $D_u(t = 0)$. We use the notation $D_u(t)$ to emphasize that the facility number changes with time during the simulation, while ρ_u and m_u are always fixed as soon as they are distributed. Since the area of all u is 1, ρ_u and D_u are exactly the population density and facility density, respectively. The number of people visiting facility i in place u, denoted as n_{iu}, is calculated as $n_{iu} = \rho_u/D_u$, while the average distance for these people to travel to facility i, denoted as $\langle r_{iu}\rangle$, is calculated as $\langle r_{iu}\rangle \sim \sqrt{s_{iu}} = 1/\sqrt{D_u}$ [18]. Then we can determine the benefit $c_{iu} = n_{iu}\langle r_{iu}\rangle^{\beta(m_u)} \sim \rho_u/D_u^{1+\beta(m_u)/2}$ for every facility i in the unit square u. Note that this quantity only depends on place u and is equal for any i within this unit square, so we can replace c_{iu} by the notation c_u. At each time step of our simulation, we calculate the c_u for each place according to the current $D_u(t)$. Then we eliminate a facility in the place with the lowest c_u and create one in the unit square with the highest c_u. This procedure is repeated until the system reaches its steady state, at which the relation between D and ρ is stable, as are the exponents.

We test this coarse-grained method by repeating the simulation in Ref. [1], i.e. setting $\beta(m) = constant$ as Eq. (1). We find that the method behaves in the same way as the simulation in Ref. [1], and reproduces all their results. In our own simulation, we have 200 different unit squares. And we set $p(\rho) = 1/\rho$ with $\rho \in [0, 5000]$ and $p(m) = 1/300$ with $m \in [0, 300]$. The initial distribution of facilities is also uniform, with $D(t = 0) \in [0, 1000]$. We choose a power-law distribution of ρ because (i) it coincides with the real population distribution which is observed to be heavy tailed; (ii) the power-law formula leads to uniformly distributed data points on a log–log plot, which gives a better visualization. Note that other distributions do not change the simulation results qualitatively. Using these parameters, we obtain $\beta(m) = 1 - (m/300)^{\mu}$.

When the simulation reaches its steady state, we measure various quantities and their relations just as done for the empirical data. Note that the classification here is carried out according to the value of m. Specifically, we divide the value of m (not the logarithmic value of m) into four equal intervals. Those unit squares u whose m_u lies in a common interval belong to the same class. Although there is a small difference from what we have done for the empirical data, this does not change our conclusion at all.

Appendix B. Analytical calculations

Since $\alpha = \frac{\mathrm{d}\langle \ln(D_i)\rangle_{m_i}}{\mathrm{d}\ln(\rho)} = \langle \alpha_i \rangle_{m_i}$, the macroscopic exponent is exactly an average of α_i over all possible m_i, i.e. $\alpha = \int_{m_{\min}}^{m_{\max}} p(m)\alpha(m)\mathrm{d}m$, where $\alpha(m) = 2/(\beta(m) + 2)$. Substituting the corresponding parameters and equation, we have

$$\alpha(\mu) = \int_0^1 \frac{2}{3 - x^\mu}\mathrm{d}x. \tag{B.1}$$

This function gives a good agreement with the simulation, as plotted in Fig. 5. In particular, for $\mu = 0, 0.2, 0.5, 1, 2, 5, 10$, Eq. (B.1) gives $\alpha = 1, 0.926, 0.866, 0.81, 0.76, 0.712, 0.692$, which is consistent with the simulation data $\alpha = 1, 0.91, 0.85, 0.8, 0.755, 0.71, 0.68$ in Fig. 5. We can also calculate the multi-scaling exponent for each class. The calculation is given by $\alpha(C) = \int_{m_{\min}}^{m_{\max}} p(m|C)\alpha(m)\mathrm{d}m$, where $p(m|C)$ is the conditional probability density under class C and is equal to $1/75$ in our simulation. $m_{\min}$ and $m_{\max}$ here are related to the specific class C. We calculate the multi-scaling exponents for $\mu = 1$ and have the theoretical result 0.69, 0.76, 0.84, 0.94 for class 1, 2, 3, 4, respectively, which is also consistent with the simulation data 0.68, 0.74, 0.81, 0.91 in the inset of Fig. 4.

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[17] In this case the scaling relation for a specific m_i is $D = m_i \frac{N_f}{\langle m\rangle N_p}\rho$. Therefore the scaling exponent $\alpha = 1$ regardless of m_i.
[18] Since we assume that all the situations in one unit square are identical, both the number of people and the area belonging to a facility are equal everywhere, leading n_i and s_i to be equal to their corresponding average, i.e., ρ_u/D_u and $1/D_u$, respectively.