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Double power-law distribution in spatial network induced by cost constraints

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Abstract – We investigate the scaling laws of the degree distribution in an evolving spatial network where the long-range links of a node are subject to a cost constraint. The constraint can cause a discontinuous reduction in the length of the links to be attached to the node once the node reaches some critical degree k_c. We show that this effect can result in an abrupt change in the attachment probability and consequently induces a double power-law degree distribution. We derive the distribution analytically for the homogeneous constraint and demonstrate a consistent result for the heterogeneous one. Our model finds a robust connection between the double power law and the spatial constraint and offers a plausible explanation of the common occurrence of the distribution in airline networks.

Introduction. – Scaling laws in complex systems represent the focus of many studies in different fields of research with important scientific applications [1–12]. While generally the scaling law manifests itself as a single power-law distribution, its behavior in some cases can change abruptly at some critical point and forms a second power-law segment at the tail [13]. Such distribution, which displays two different scaling laws, is called double power law, whose cumulative distribution, namely the probability that a random variable is larger than a specific value k, is given by

$$P(k) \sim \begin{cases} k^{-\gamma_1}, & k \le k_c, \\ k^{-\gamma_2}, & k > k_c, \end{cases} \tag{1}$$

where γ_1 and γ_2 are two exponents and k_c is the transition point. Recent empirical studies have witnessed this distribution in a variety of complex systems. The examples include the degree of airline networks and word networks [14–23], the size of computer files and forest fires [24,25], the number of words in Homer's epic poem and the number of hits received by websites [25], the severity of tornadoes and terrorist attacks [25], individual incomes [26–29], human collaboration behavior and the length of street segment [30–32]. In all these cases, the exponent γ_2 is found to be larger than γ_1, showing a sharper gradient and a narrower range than a single power law.

In network systems, spatial constraint is considered as a major factor that can affect the scaling-law behavior of the degree distribution [33]. It originates from maintenance cost, transmission delay or energy dissipation and usually manifests itself as the preference of spatially short links. This preference competes with the preferential attachment rule and suppress the emergence of large degree. Evidences from the real networks have demonstrated a crossover from the skewed power law to the fast decaying exponential one with the increase of the constraint strength [34–39]. The double power law, which decays at an intermediate rate, is found to commonly occur in the airline networks that has just the intermediate strength [16–23], which indicates a connection between the spatial constraint and the particular distribution. To test the conjecture, one can set up a network model and inspect the degree distribution while adjusting the constraint strength. With lots of work dedicated to the issue [40–46], scientists are able to reproduce most of the distributions, yet fail to realize the double power law during the adjustment. Instead, several models with no spatial relevance [13–15,26], though insufficiently

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validated, seem to reach the goal, raising the doubt of the truth of the conjecture.

In this paper, we show that by imposing a cost constraint on each node's long-range links, the double power law can indeed arise from the spatial effect. Our inspiration comes from a hot topic in network science, called the navigation problem [47–51]. The network architecture in the relevant issue is built with costless local links and long-range connections whose cost are proportional to their length. The constraint on these costs, as a natural consequence of the limited resources in general case, plays a key role on the navigation behavior [50,51]. However, since these studies take no interest in the evolution dynamics, the realization of the network is always implemented by laying a substrate with local links in the first place, then supplementing additional long-range connections. We find that if this order is reversed, the cost constraint can even affect the scaling law and the double power-law degree distribution will emerge spontaneously.

The rest of the paper is organized as follow. In the following section, we present the detailed description of our model. In the third section, we analyze the scaling-law property of the degree distribution for the homogeneous and the heterogeneous constraint respectively and present both analytical and simulation results. In the fourth section, we draw the conclusion and make a discussion of the implication of our model to the airline network.

Model description. – Consider a city that plans to improve its reachability or an individual who wishes to expand his social circle, their final connectivity depends on their own attributes such as population capacity, energy, or relevance as termed in some literature [52], which can be uniformly characterized as a cost constraint quantified by a budget. In a spatial network, the cost of maintaining a long-range link is supposed to be proportional to its length. In contrast, if the length decreases to a certain degree, the cost could be negligible, just as our daily encounters with neighbors, which requires few expenditures of time and effort. This assumption is applied in the navigation problem [47–51] and is supported by an empirical observation on U.S. airport network [53], where the probability of a connection is inferred to decay only when the distance between airports exceeds a critical value. Therefore, the budget of a node is mainly consumed by its long-range links. As a consequence, a new coming node can only connect with others who either have budget margin or are located just nearby, which represents the essence of our model.

Our network evolves by continuously adding nodes and edges on a square plane with periodic boundary. For each new-coming node t, we assign an initial budget B_t and a proportionate length L_t, which quantify the total length and the maximum length of its long-range links respectively. To give a definition of the long-range links, we introduce a universal length $r_0(< \min L_t)$ as a division, below which the links are considered to be costless. The detailed evolution rules are described as follows:

a) At each time step t, a new node t with parameters B_t and L_t is added to the network and is placed at a randomly chosen position.

b) The node t selects its target i within the set G_t by probability $\frac{k_i}{\sum_{i\in G_t} k_i}$ to create a new link, where k_i is the degree of node i belonging to G_t, and G_t is composed of all the nodes who either are located within r_0 from t or are located within their maximum length from t and meanwhile have budget margin, *i.e.*, $B_i > 0$.

c) If the length of the new created link l_t is larger than r_0, refresh the budget of node i by $B_i = B_i - l_t$ and repeat the whole process.

Note that rule c) will never be implemented when $B_i < 0$ because no links with length larger than r_0 can be created in this situation according to rule b). Rule c) can also be modified to further include the reduction of the budget of the new-coming node, *i.e.*, B_t. However, the result turns out to be no significantly different. See the Supplementary Material `Supplementarymaterial.pdf` (SM) for a discussion. In the initial stage of the evolution, there are only a few nodes distributed sparsely on the plane. It is very likely that the set G_t of some new-coming nodes is empty. When this occurs, we just connect the node t with a randomly chosen target to keep the network connected and continue the procedure. With the increase of the node density, the empty sets will no longer occur. The initial random effect vanishes so quickly, as evaluated in the SM, and will not affect the final structure of our model. In this paper, we only focus on the condition of sufficient node density.

Scaling law of degree distribution. – For an existing node i, the event that it belongs to G_t occurs only when it is located sufficiently close to the new-coming node t. The critical distance, which is determined by either L_i or r_0 according to rule b), defines a circular influence zone of i with the area denoted by S_i, which can also be interpreted as the strength of the spatial constraint. Let A be the area of the whole plane, the probability of the event is exactly the probability that the node t is located within the influence zone, which is given by S_i/A. Hence, the probability of the new-coming node to connect to the node i is

$$p_i = \frac{S_i}{A}\frac{k_i}{\sum\limits_{i\in G_t} k_i}. \tag{2}$$

From the view of the mean-field theory, each node i in the network contributes its degree weighted by $\frac{S_i}{A}$ to $\sum_{i\in G_t} k_i$, giving $\frac{1}{A}\sum_i S_i k_i$ in all. Then eq. (2) is rewritten as

$$p_i = \frac{S_i k_i}{\sum\limits_i S_i k_i}, \tag{3}$$

and the dynamical equation of degree k_i can be derived

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = \frac{S_i k_i}{\sum_i S_i k_i}. \tag{4}$$

Homogeneous cost constraint. Let the initial budget $B_i = B$ and correspondingly $L_i = L$ for all nodes, the area of the influence zone of a node i at the early stage of its evolution takes $S_i = \pi L^2$ because the sufficient budget allows it to access any place within distance L. However, once the long-range links exhaust the budget, *i.e.*, $B_i < 0$, only local links are available in the subsequent evolution, resulting in a discontinuous reduction in link length and a new influence zone with area $S_i = \pi r_0^2$. As a consequence, both the connection probability (eq. (3)) and the evolution dynamics (eq. (4)) changes abruptly at this time. To evaluate when this critical event occurs, recall that we assume r_0 is small, therefore, the probability of a new link being shorter than r_0, which can be calculated by r_0^2/L^2, is generally negligible when $B_i > 0$. In other words, almost all the links created at this stage are long-range connections, whose length scales as $O(L)$. Hence, the critical event is expected to occur when the node i reaches the transition degree

$$k_c \sim \frac{B}{L}. \tag{5}$$

Equation (5) indicates that eq. (4) can be specifically expressed as two sub-equations in terms of the node degree,

$$\begin{aligned} \frac{\mathrm{d}k_i}{\mathrm{d}t} &= \frac{L^2 k_i}{L^2 \sum\limits_{i|k_i \le k_c} k_i + r_0^2 \sum\limits_{i|k_i > k_c} k_i}, \quad k \le k_c, \\ \frac{\mathrm{d}k_i}{\mathrm{d}t} &= \frac{r_0^2 k_i}{L^2 \sum\limits_{i|k_i \le k_c} k_i + r_0^2 \sum\limits_{i|k_i > k_c} k_i}, \quad k_i > k_c, \end{aligned} \tag{6}$$

where the notations $\sum\limits_{i|k_i \le k_c}$ and $\sum\limits_{i|k_i > k_c}$ represent the sum over i with $k_i \le k_c$ and $k_i > k_c$, respectively. Let $f(t) = L^2 \sum\limits_{i|k_i \le k_c} k_i + r_0^2 \sum\limits_{i|k_i > k_c} k_i$, it is bounded by $2r_0^2 t \le f(t) \le 2L^2 t$, which suggests an ansatz $f(t) = Ct$. To derive a self-consistent solution, we substitute the ansatz into eq. (6) and find

$$\begin{aligned} k_i(t) &= \left(\frac{t}{i}\right)^{\frac{L^2}{C}}, \qquad k_i \le k_c, \\ k_i(t) &= k_c \left(\frac{t}{t_c}\right)^{\frac{r_0^2}{C}}, \quad k_i > k_c, \end{aligned} \tag{7}$$

where t_c is the time when the node degree reaches k_c, or equivalently, when the budget is exhausted. It can be obtained from the first equation in eq. (7) by letting $k_i(t) = k_c$, which gives $t_c = i k_c^{\frac{C}{L^2}}$. Substituting eq. (7) into the definition formula of $f(t)$ and applying a continuous approach, we have

$$f(t) = L^2 \int_{i_c}^{t} \left(\frac{t}{i}\right)^{\frac{L^2}{C}} \mathrm{d}i + r_0^2 \int_{1}^{i_c} k_c \left(\frac{t}{t_c}\right)^{\frac{r_0^2}{C}} \mathrm{d}i, \tag{8}$$

where $i_c = t k_c^{-\frac{C}{L^2}}$ represents the identity of the node with degree k_c at time t. If eq. (7) is a true solution of eq. (6), the calculation of eq. (8) must satisfy $f(t) = Ct$. The self-consistent condition allows us to solve C from the following equation:

$$C(r_0^2 - L^2) k_c^{1-\frac{C}{L^2}} = (C - 2L^2)(C - r_0^2), \tag{9}$$

and complete the whole derivation of $k_i(t)$. Then a standard treatment in ref. [3] helps to derive the cumulative degree distribution. We find

$$P(k) = \begin{cases} k^{-\gamma_1}, & k \le k_c, \\ \lambda k^{-\gamma_2}, & k > k_c, \end{cases} \tag{10}$$

where $\gamma_1 = \frac{C}{L^2}$, $\gamma_2 = \frac{C}{r_0^2}$ and $\lambda = k_c^{\gamma_2 - \gamma_1}$. The distribution is a double power law with exponent $\gamma_2 > \gamma_1$ and a transition point at k_c. It is notable that the strength of the spatial constraint, quantified by L^2 and r_0^2, serves only as the tunable parameter of the scaling exponent. Adjusting its magnitude will vary the exponent but cannot create more scaling-law regimes. This helps to understand why the double power law cannot be produced by simply adjusting the parameters related to the constraint strength as was done in many existing models [40–46], because no matter how we adjust it, there is still only one particular type of strength. Instead, it is the abrupt change of the constraint strength during the evolution that introduces a new scaling behavior and causes the emergence of double power law, and the cost and the budget offers a corresponding reasonable explanation of how this abrupt change can occur spontaneously.

We test our analytical results by performing the Monte Carlo simulations. Figure 1 shows the results of the cumulative degree distribution for different r_0. We find all of them display the double power-law behaviors as the theoretical prediction. Careful examination indicates a little deviation of k_c for blue data points, which is to be expected because r_0 in this case is not small enough. We also observe that the slopes of the second power-law segments present a dependence on the parameters but the effect seems insignificant for the first ones. To assess the quantitative accuracy of our analytical analysis, we measure the power-law exponents γ_1, γ_2 and the transition degree k_c (see the SM for the methodology of our data analysis), and plot them *vs.* L^2 and B, respectively, as shown in fig. 2 and its inset. The theoretical relation between γ_1, γ_2 and L^2 can be deduced by replacing C in eq. (9) by the relation $\gamma_1 = \frac{C}{L^2}$, $\gamma_2 = \frac{C}{r_0^2}$ and replacing k_c by $\frac{B}{L}$. The results (dot-dashed lines) show nice agreements with the measurements. The different sensitivity

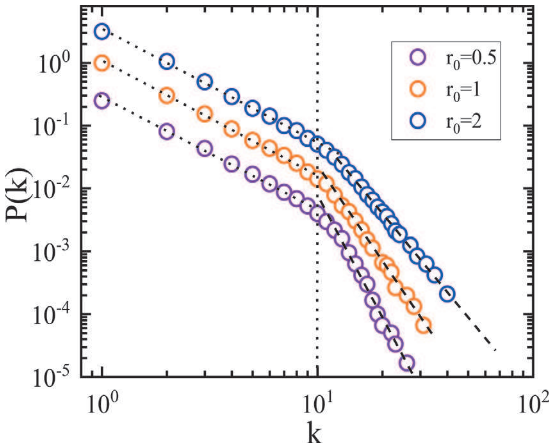

Fig. 1: The cumulative degree distributions for three different r_0. The simulations are performed under the parameters $A = 100$, $B = 20$ and $L = 3$. The network size reaches 1.5×10^4 nodes. The dotted lines and the dashed lines represent the guidelines for the slopes of the upper tails and lower tails, respectively. The vertical dotted line is a guide to the transition point k_c. Note that the blue and the purple points are shifted upward and downward, respectively, for better visualization.

of the two exponents to the parameters is consistent with our previous observation in fig. 1. The measured transition degree k_c shows a linear relation with the budget B, justifying the validity of eq. (5).

Heterogeneous cost constraint. Generally, a node with high initial budget (such as a city with high economic level) presents a proportionate large influence and reach. The proportionate effect indicates the fraction B_i/L_i takes equal value for all nodes despite their diverse scales of long-range links. In other words, we can still expect an abrupt change of both the link length and the dynamics of $k_i(t)$ at a universal transition point $k_c \sim B_i/L_i$. As a consequence, eq. (4) can be written as

$$\begin{aligned} \frac{\mathrm{d}k_i}{\mathrm{d}t} &= \frac{L_i^2 k_i}{\sum\limits_{i|k_i \le k_c} L_i^2 k_i + r_0^2 \sum\limits_{i|k_i > k_c} k_i}, \quad k \le k_c, \\ \frac{\mathrm{d}k_i}{\mathrm{d}t} &= \frac{r_0^2 k_i}{\sum\limits_{i|k_i \le k_c} L_i^2 k_i + r_0^2 \sum\limits_{i|k_i > k_c} k_i}, \quad k_i > k_c, \end{aligned} \tag{11}$$

where L_i^2 can be considered to be taken from some distribution $\rho(L_i^2)$ with finite support ($L_i \le \sqrt{2A}$). Let $f(t) = \sum\limits_{i|k_i \le k_c} L_i^2 k_i + r_0^2 \sum\limits_{i|k_i > k_c} k_i$ and consider the ansatz $f(t) = Ct$, we can solve eq. (11) by applying the similar self-consistent approach as have done previously. Note that the calculation of the first term in $f(t)$ in the present heterogeneous case should take an average over all possible L_i^2 subject to $\rho(L_i^2)$. For brevity, we omit the specific derivation of C as its value is not essential for the following discussion.

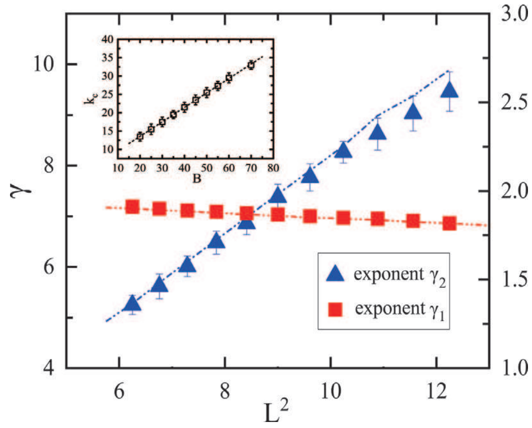

Fig. 2: The measured power-law exponents of both tails *vs.* L^2. The simulations are performed under the parameters $A = 100$, $B = 20$ and $r_0 = 1.5$. The network size reaches 1.5×10^4 nodes. The left and the right axes correspond to the scale of γ_2 and γ_1, respectively. Each data point is an average over 20 independent runs. The dot-dashed lines are the theoretical predictions of the relation between γ_1(red), γ_2(blue) and L^2. Note that the error bar of γ_1 is smaller than the symbol. Inset: the relation between the transition degree k_c and the budget B. The parameters are $A = 100$, $r_0 = 1.5$, $L = 3$, $B \in [20, 70]$ and the network size of 1.5×10^4. The dashed line is a guide to eyes.

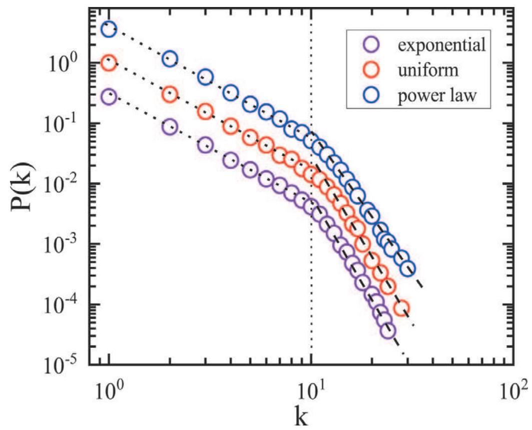

Fig. 3: The cumulative degree distributions for three different budget distributions, including uniform distribution, exponential distribution (with exponent −0.3) and power-law distribution (with exponent −3). Other parameters includes the plane area $A = 100$, the maximum budget $B_{max} = 40$, the minimum budget $B_{min} = 15$, the scaling factor $\frac{B}{L} = 7$ and $r_0 = 0.5$. The network size reaches 1.5×10^4 nodes. The dotted lines and the dashed lines represent the guidelines for the slopes of the upper tails and lower tails, respectively. The vertical dotted line is a guide to the transition point k_c. Note that the blue and the purple points are shifted upward and downward, respectively, for better visualization.

When substituting $f(t) = Ct$ back into eq. (11), we find the first equation in eq. (11) is the same with the dynamics of the fitness model [54]. The pioneering relevant work [52,54–56] has demonstrated that the resulting degree distribution is a mixture of the multiple scaling laws weighted by $\rho(L_i^2)$ and usually can be approximated by a generalized power law with the exponent γ_1 bounded by C/L_{max}^2 and C/L_{min}^2. On the other hand, the second equation corresponds to a power law with exponent C/r_0^2, which is larger than the upper bound of γ_1. We therefore show that the double power-law degree distribution can emerge in very general cases and indicate its robust connection with the cost constraint and the induced discontinuous reduction of link length. In fig. 3, we plot the cumulative degree distributions of our simulations for three different budget distributions. All of them present double power-law behaviors as the theoretical prediction.

Conclusion. – We propose an evolving spatial network model where the long-range connections of a node are subject to a cost constraint while the local links are created without stint. A new-coming node can only connect with others who either have budget margin or are located just nearby. When the budget of a node is exhausted, with only local links available, its subsequent connections get an discontinuous decrease in length. We find that this effect can induce an abrupt change in the dynamics of the node's degree and produces a new scaling law at the tail of the degree distribution, resulting in the emergence of the double power law. We derive the power-law exponents and the transition point analytically for the homogeneous constraint, showing quantitatively how the spatial effect affects the scaling-law behavior, and demonstrate a consistent result for the heterogeneous one, indicating the robustness and universality of our theory.

Our model has particular implications to the airline networks. In contrast to land transportation that is strongly constrained by geography, the constraint of constructing a flight route to a city is more related to the economic level of the city or the handling capacity of the corresponding airport, which is in agreement with one of the key mechanisms in our model, the budget constraint on nodes. On the other hand, the accessibility of a city benefits more from those distant flight routes, which are usually costly for other transportation and thus irreplaceable. In this sense, establishing the distant flight route in priority seems intuitively a more efficient and reasonable choice for a city, particularly under the limited budget. This implies that the flight routes launched earlier are expected to be statistically longer than those later. A plausible evidence for the inference is available in ref. [57], where the estimated distance coefficient in gravity model shows an increase when comparing the airline data of 1996 and 2004, which is consistent with the consequence of the increase of short links in number and thus indicates the reduction of the length of new routes. If the reduction occurs in the same way with our model, namely abruptly from one length to the other, we would expect two major scales dominating the link length, which just corresponds with the bimodal distribution found in airline networks [58]. Though a rigorous validation needs more evidences, the facts available at present indicate that the key mechanisms of our model that are responsible for the emergence of double power law might be related to the nature of the airline system and govern its underlying dynamics, which offers a plausible explanation of why such distribution is so common in this kind of network. There are some other practical factors that our model has not considered, including the node position heterogeneity, the exclusion of too short links and the unknown budget distribution to be evaluated. The application of our model to the real airline network must overcome these challenges. We have made proper discussions of these issues in the SM and it turns out that none of them would change our results significantly.

* * *

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